The Empire Open Math Contest - June 2026 Casual

Official Solutions

Full solutions to all fifteen problems. Each final answer is boxed.

Problem 1. $\triangle ABC$ is a right triangle with $A = (2, 3)$, $B = (5, 7)$, and $\angle B = 90°$. Given that the y-coordinate of C is $-\dfrac{1}{5}$, find the x-coordinate of C, expressed as a fraction in lowest terms. This fraction is written as $\frac{p}{q}$, where p, q are positive integers and the greatest common factor between them is 1. Report your answer as $p + q$.

Solution. Two lines that meet at a right angle have slopes that are negative reciprocals (their slopes multiply to -1).

The slope of BA, from $B(5, 7)$ to $A(2, 3)$, is

$$\frac{3-7}{2-5} = \frac{-4}{-3} = \frac{4}{3}.$$

Since BC meets BA at a right angle at B, the slope of BC is the negative reciprocal, $-\frac{3}{4}$. The slope of BC, from $B(5, 7)$ to $C\left(x, -\frac{1}{5}\right)$, is

$$\frac{-\frac{1}{5}-7}{x-5} = \frac{-\frac{36}{5}}{x-5}.$$

Set the two equal:

$$\frac{-\frac{36}{5}}{x-5} = -\frac{3}{4}.$$

Cross-multiplying gives $-\frac{36}{5} \cdot 4 = -3(x-5)$, so $-\frac{144}{5} = -3(x-5)$. Dividing both sides by -3: $\frac{48}{5} = x - 5$. Therefore

$$x = 5 + \frac{48}{5} = \frac{25}{5} + \frac{48}{5} = \frac{73}{5}.$$

Adding $73 + 5$:

Answer: $\boxed{78}$

Problem 2. You arrive at Penn Station and step outside and notice a peculiar smell. You look up and notice that an analog clock reads 4:20. What is the smaller angle, in degrees, between the hour hand and the minute hand?

Solution. The minute hand points straight at the 4, which is 20 minutes around the clock. A full circle is $360°$ split into 60 minutes, so each minute is $6°$; the minute hand is therefore at $20 \times 6° = 120°$ from the top.

The hour hand moves too. It travels 30° per hour ($360^\circ \div 12$), which is 0.5° per minute. By 4:20 it has gone 4 hours and 20 minutes, so it sits at $4 \times 30^\circ + 20 \times 0.5^\circ = 120^\circ + 10^\circ = 130^\circ$. The hands are $130^\circ - 120^\circ = 10^\circ$ apart.

Answer: $\boxed{10^\circ}$

Problem 3. The council of Pigeons convenes in Central Park to decide their next target. There are 12 pigeon-lords who each control different sections of the city. Before starting the meeting every pigeon-lord dabs up (shakes hands) with every other pigeon-lord. How how many dabs (handshakes) occur?

Solution. Picture the 12 people. Each one shakes hands with the other 11, which looks like $12 \times 11 = 132$ handshakes. But every handshake involves *two* people, so each one has been counted twice, once from each person's point of view. To fix the double-counting, divide by 2:

$$\frac{12 \times 11}{2} = \frac{132}{2} = 66.$$

(This is exactly the reasoning behind the "choose 2" shortcut, $\frac{n(n-1)}{2}$.)

Answer: $\boxed{66}$

Problem 4. You're riding the 6 train and encounter a delay. To your fortunate surprise you actually have signal but to your dismay every time you refresh your phone on Google Maps to see your estimated time of arrival at Grand Central the time goes up...

The sequence of times is given by T_n as follows:

$$T_1 = 3$$

and

$$T_{n+1} = 2T_n + 1.$$

What is T_5?

Solution. Write out the first few terms:

$$a_1 = 3, \quad a_2 = 2(3) + 1 = 7, \quad a_3 = 2(7) + 1 = 15, \quad a_4 = 2(15) + 1 = 31.$$

Each term is one less than a power of 2: $3 = 4 - 1$, $7 = 8 - 1$, $15 = 16 - 1$, $31 = 32 - 1$, that is $2^2 - 1$, $2^3 - 1$, $2^4 - 1$, $2^5 - 1$. The pattern is $a_n = 2^{n+1} - 1$.

(Why it keeps working: adding 1 to each term gives $4, 8, 16, 32, \ldots$, a doubling pattern, because $a_{n+1} + 1 = 2a_n + 2 = 2(a_n + 1)$.) So $a_{2026} = 2^{2027} - 1$.

Answer: $\boxed{2^{2027} - 1}$

Problem 5. A bag contains 3 blue balls and 5 orange balls (Go Knicks!). Two balls are drawn without replacement (meaning after the ball is taken out of the bag, another copy of that ball miracuously appears inside the bag). The probability that they are the same color is

$$\frac{a}{b},$$

where a and b are relatively prime positive integers. What is $a+b$?

Solution. Draw the two balls one after the other.

Both blue: the first ball is blue 3 times out of 8; with one blue gone, the second is blue 2 times out of the remaining 7. Multiply: $\frac{3}{8}\cdot\frac{2}{7}=\frac{6}{56}$.

Both orange: similarly $\frac{5}{8}\cdot\frac{4}{7}=\frac{20}{56}$.

"Same color" means one case *or* the other, so we add them:

$$\frac{6}{56}+\frac{20}{56}=\frac{26}{56}=\frac{13}{28}.$$

This is already in lowest terms, so $a+b=13+28=41$.

Answer: $\boxed{41}$

Due to a typo we also accepted the case when the balls were drawn with replacement. In that case you consider the case when both balls are blue and both balls are orange that are drawn from the bag. The sum of case one and case two is:

$$\frac{3}{8}\cdot\frac{3}{8}+\frac{5}{8}\cdot\frac{5}{8}=\frac{9}{64}+\frac{25}{64}=\frac{34}{64}=\frac{17}{32}\Rightarrow 17+32=49 \qquad (1)$$

Answer: $\boxed{49}$

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Problem 6. A hot dog costs \$2 in Central Park and a pretzel costs \$3. Suppose your net worth is \$37. How many ways can spend your entire net worth in Central Park on hot dogs and pretzels?

Solution. Let x be the number of hot dogs, y be the number of pretzels.

- Positive: $3y<37$, so $y\leq 12$.
- Even: since 37 is odd, $3y$ must be odd, which forces y to be odd.

The odd values $y=1,3,5,7,9,11$ each give a positive x (namely $17,14,11,8,5,2$). That is 6 solutions.

Answer: $\boxed{6}$

Problem 7. You always forget which street your friend lives on in Manhattan, but you do remember it's below 100th Street. You call to ask, but your friend is annoyed and will only tell you that that the street number leaves a remainder of 2 when divided by 5 and a remainder of 3 when divided by 7. What is the sum of all possible street numbers?

Solution. Start with the numbers that leave remainder 3 when divided by 7: begin at 3 and keep adding 7,

$$3,\ 10,\ 17,\ 24,\ 31,\ \ldots$$

We also need remainder 2 when divided by 5 — and a number leaves remainder 2 when divided by 5 exactly when it *ends in* 2 *or* 7. The first number in the list ending in 2 or 7 is 17.

Once we have one answer, adding 35 ($= 5 \times 7$) keeps both remainders the same, so the next answers are $17 + 35 = 52$ and $52 + 35 = 87$. The next would be 122, which is too big. Their sum is $17 + 52 + 87 = 156$.

Answer: $\boxed{156}$

Problem 8. Governor Hochul has assigned you to be the head of the New York State Lottery. It has been decided that the answer to this year's lottery is a 3 digit number. While deciding what the number should be, Mr. Crabs walks out of the TV and lets you know that he insists that the lottery should be rigged. He desires that the number ends with a 5 and also has to be divisible by 9. How many possible lottery numbers are there that can meet Governor Hochul and Mr. Crabs's constraints?

Solution. A number is divisible by 9 exactly when its digits add up to a multiple of 9. Our number looks like

$$(\text{first})(\text{second})\,5,$$

where the two front digits come from 1–9, must be different, and cannot be 5. So we need (first) + (second) + 5 to be a multiple of 9.

The two front digits add up to somewhere between $1 + 2 = 3$ and $8 + 9 = 17$, so the full three-digit total lands between 8 and 22. The only multiples of 9 in that range are 9 and 18, which means the two front digits add to 4 or to 13:

- add to 4: only 1 and 3;
- add to 13: 4 and 9, or 6 and 7 (not 5 and 8, since 5 is taken).

Each pair can be ordered two ways, giving $135, 315, 495, 945, 675, 765$ — 6 numbers in all.

Answer: $\boxed{6}$

Problem 9. The year is 2090 and the term "rizz" is cool again. These days however kids use telepathic brain computer interfaces to keep the meme cool they don't even articulate what magic rizz-number is. You have been tasked with finding the number and have just received the following information from discord: $s(n)$ be the sum of the digits of the two-digit number n. If

$$n + s(n) = 61,$$

what is n?

Solution. A two-digit number with tens digit a and units digit b equals $10a + b$, and its digit sum is $s(n) = a + b$. So the equation becomes

$$\underbrace{(10a + b)}_{n} + \underbrace{(a + b)}_{s(n)} = 11a + 2b = 61.$$

Here a is a digit from 1 to 9 and b from 0 to 9. Notice $2b$ is always even, but 61 is odd, so $11a$ must be odd — and that happens only when a is odd. Test the odd choices:

- $a = 1$: $11 + 2b = 61 \Rightarrow 2b = 50 \Rightarrow b = 25$ (too big);
- $a = 3$: $33 + 2b = 61 \Rightarrow 2b = 28 \Rightarrow b = 14$ (too big);
- $a = 5$: $55 + 2b = 61 \Rightarrow 2b = 6 \Rightarrow b = 3$ ✓;

- $a = 7$: 77 is already past 61 (too big).

Only $a = 5$, $b = 3$ works, so $n = 53$. Check: $53 + (5 + 3) = 61$. ✓

Answer: $\boxed{53}$

Problem 10. Your realtor has helped have find an excellent deal on a rectangular studio rental listing. Unfortunately the listing has no photos, and only states that the perimeter is 50 feet and the area is 150 square feet. What is the length of the longer side of the studio?

Solution. Let a and b be the two side lengths. We are given the perimeter and the area so that $2(a + b) = 50$ and $ab = 150$. From the first equation, $a + b = 25$ so $b = 25 - a$. Plugging that in to the second equation: $a(25 - a) = 25 - a^2 = 150$. Simplifying, we get $a^2 - 25a + 150 = 0 \implies (a - 10)(a - 15) = 0$.

The two side lengths are 10 and 15, of which the larger is 15.

Answer: $\boxed{15}$

Problem 11. Bailey has a Brownstone in Brooklyn. She's looking at her view but unfortunately some developers have put some eyesore skyer scraper right in the way. Luckily on her block the skyscrapers are only on points (a, b) where $ab = 12$. We want to count how many corners she can see from her house so we are asking:

How many ordered pairs (a, b) of positive integers with $1 \le a, b \le 30$ have the property that ab is divisible by 12?

Solution. Since $12 = 4 \times 3$, we need the product ab to be divisible by both 4 and 3. There are $30 \times 30 = 900$ pairs in total. It is easier to count the *bad* pairs (where ab is *not* divisible by 12) and subtract. A pair is bad if ab misses the factor of 4, or misses the factor of 3, or both.

Missing the 4. For ab to be divisible by 4, the two numbers together need at least two factors of 2. Sort 1–30 by how many 2s each has:

- 15 odd numbers (zero 2s);
- 8 numbers equal to $2 \times$ (odd): $2, 6, 10, 14, 18, 22, 26, 30$ (one 2 each);
- 7 multiples of 4: $4, 8, 12, 16, 20, 24, 28$ (two or more 2s).

The product misses the 4 only when the pair has fewer than two 2s between them — both odd, or one odd with one "single-2" number. Counting: both odd is $15 \times 15 = 225$; odd-with-single-2 (in either order) is $15 \times 8 + 8 \times 15 = 240$. Total $= 465$.

Missing the 3. This means neither number is a multiple of 3. Of 1–30, ten are multiples of 3, so 20 are not; pairs from those 20 number $20 \times 20 = 400$.

Counted twice. Pairs that miss *both* were included in each tally above, so they got counted twice — subtract them once. Among the 20 non-multiples of 3, sorting by 2s gives 10 odd, 5 single-2 $(2, 10, 14, 22, 26)$, and 5 multiples of 4 $(4, 8, 16, 20, 28)$. The ones that also miss the 4: both odd is $10 \times 10 = 100$, odd-with-single-2 is $10 \times 5 + 5 \times 10 = 100$; total 200.

So the bad pairs number $465 + 400 - 200 = 665$, and the good pairs number

$$900 - 665 = 235.$$

Answer: $\boxed{235}$

Problem 12. The Manhattan street grid forms perfect blocks, each 250 feet long and 900 feet wide. A pigeon flies in a straight line from the bottom-right corner of one block to the top left corner of the block above it. The distance the pigeon flew can be expressed as $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime. What is $a+b$?

Solution. Two adjacent blocks line up into one big rectangle. Putting them together so the 250-foot sides combine gives a rectangle $250 + 250 = 500$ feet on one side and 900 feet on the other. The pigeon flies corner to corner along the diagonal, so by the Pythagorean theorem the distance is

$$\sqrt{500^2 + 900^2} = \sqrt{250000 + 810000} = \sqrt{1{,}060{,}000} = \sqrt{100^2 \cdot 106} = 100\sqrt{106}.$$

Since $106 = 2 \times 53$ has no repeated prime factor, $b = 106$ is square-free, and $a+b = 100+106 = 206$.

Answer: $\boxed{206}$

Problem 13. The number 24 is an abundant number. Its sum of divisors is bigger than it. We like lets define a "chill" number as a number whose sum of divisors is an abundant number. Consider the 24-"chill" numbers these are the numbers whose sum of all positive divisors is exactly 24. Find the sum of all chill numbers.

Solution. We want numbers whose divisors add up to 24. Every number N has both 1 and N itself as divisors, so its divisors already add up to at least $N+1$. For the total to be 24 we need $N+1 \leq 24$, that is $N \leq 23$ — so we only have to check the numbers up to 23. Three of them work:

- 14: divisors $1, 2, 7, 14$, and $1 + 2 + 7 + 14 = 24$ ✓;
- 15: divisors $1, 3, 5, 15$, and $1 + 3 + 5 + 15 = 24$ ✓;
- 23: it is prime, so its only divisors are 1 and 23, and $1 + 23 = 24$ ✓.

No other number from 1 to 23 reaches 24. Adding the winners: $14 + 15 + 23 = 52$.

Answer: $\boxed{52}$

Problem 14. Eh, forget it, theres no way to yassify this one. Gonna just ask it straight: How many integers less than 2027 are relatively prime to it?

Solution. First, 2027 is a prime number: testing the primes up to $\sqrt{2027} \approx 45$ (that is $2, 3, 5, \ldots, 43$), none of them divide 2027. Because it is prime, 2027 shares no common factor bigger than 1 with any smaller positive number. So every integer from 1 to 2026 is relatively prime to it — all 2026 of them.

Answer: $\boxed{2026}$

Problem 15. (Viktor Krapivin, NYCMath) Ramanujan shows up suddenly in NYC in 2026 and presents you with the mysterious infinite continued fraction

$$x = 3 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \ddots}}}$$

x can be expressed as $\sqrt{k}$. Find k.

Solution. Let y stand for the repeating part:

$$y = 6 + \cfrac{1}{6 + \cfrac{1}{6 + \ddots}}.$$

The expression sitting inside the first denominator is identical to the whole thing, so y satisfies $y = 6 + \frac{1}{y}$. Multiply through by y:

$$y^2 = 6y + 1 \quad \Longrightarrow \quad y^2 - 6y - 1 = 0.$$

By the quadratic formula, $y = \dfrac{6 \pm \sqrt{36+4}}{2} = \dfrac{6 \pm \sqrt{40}}{2} = 3 \pm \sqrt{10}$. Since y is clearly positive, $y = 3 + \sqrt{10}$. Now

$$x = y - 3 = (3 + \sqrt{10}) - 3 = \sqrt{10} \Longrightarrow k = 10.$$

Answer: $\boxed{10}$