The Empire Open Math Contest Solutions

Answer Key

21, 21, 28, 495, 4, 208, 442, 63, 9000, 1154, 91, 32, 15, 10, 12

Problem 1. The State of Liberty and you decide to go to the Hard Rock Cafe in Times Square to unwind on a Friday Night. Lady Liberty pulls out her phone to connect to the wifi but unfortunately its blocked. You ask the waiter and they point you to a sign. It's a bit dark so you rub your eyes to see clearly and in gothic chalk on a black background the following was inscribed:

Find the sum of all integers x satisfying two properties:

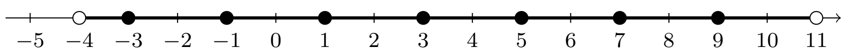
$$|2x - 7| < 15$$

and the integer remainder of the x after dividing by 8 is 1 or more technically put:

$$x^2 \equiv 1 \pmod{8}.$$

Translation. The inequality tells us which integers x are in range. The congruence tells us which of those integers have square remainder 1 modulo 8. Then we add the valid integers.

odd integers in $-4 < x < 11$



Solution. First solve the inequality:

$$|2x - 7| < 15.$$

This means

$$-15 < 2x - 7 < 15.$$

Adding 7 to all three parts gives

$$-8 < 2x < 22.$$

Dividing by 2, we get

$$-4 < x < 11.$$

So the possible integer values of x are

$$-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.$$

Now we use the modular condition:

$$x^2 \equiv 1 \pmod{8}.$$

Every odd integer has square congruent to $1 \pmod 8$. To see this, every odd integer is congruent to one of

$$1, 3, 5, 7 \pmod 8.$$

Squaring gives

$$1^2 \equiv 1, \qquad 3^2 = 9 \equiv 1,$$

$$5^2 = 25 \equiv 1, \qquad 7^2 = 49 \equiv 1 \pmod 8.$$

Even integers have square congruent to 0 or $4 \pmod 8$, not 1.

Therefore the valid values are

$$-3, -1, 1, 3, 5, 7, 9.$$

Their sum is

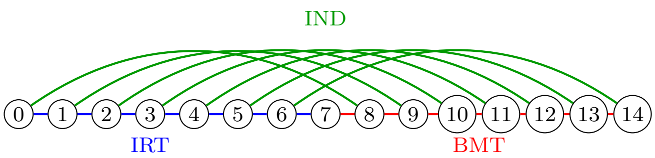
$$-3 - 1 + 1 + 3 + 5 + 7 + 9 = 21.$$

Thus the answer is

$$\boxed{21}.$$

Problem 2. (By Richard Liu, NYCMath) There are 15 subway stations across New York City. Some of them are connected by subway lines, which are operated by three divisions: the IRT, the IND, and the BMT. It is known that even if any one division suspends all of its service, it is still possible to travel between any two stations using the remaining two divisions (possibly with transfers). What is the smallest number of subway connections in the city?

Translation. Think of the subway map as a graph. Stations are points, connections are edges, and each edge has one of three colors. Deleting any one color still leaves a connected graph.



Solution. Let the numbers of IRT, IND, and BMT connections be

$$a, b, c.$$

If one division shuts down, the other two divisions must still connect all 15 stations.

A connected graph on 15 vertices needs at least 14 edges. Therefore:

$$b + c \geq 14,$$

$$a + c \geq 14,$$

$$a + b \geq 14.$$

Adding these three inequalities gives

$$2a + 2b + 2c \geq 42.$$

So

$$a + b + c \geq 21.$$

Thus at least 21 connections are necessary.

Now we show 21 is possible.

Label the stations

$$0, 1, 2, \ldots, 14.$$

Use these 7 IRT connections:

$$0-1,\ 1-2,\ 2-3,\ 3-4,\ 4-5,\ 5-6,\ 6-7.$$

Use these 7 BMT connections:

$$7-8,\ 8-9,\ 9-10,\ 10-11,\ 11-12,\ 12-13,\ 13-14.$$

Use these 7 IND connections:

$$0-8,\ 1-9,\ 2-10,\ 3-11,\ 4-12,\ 5-13,\ 6-14.$$

There are

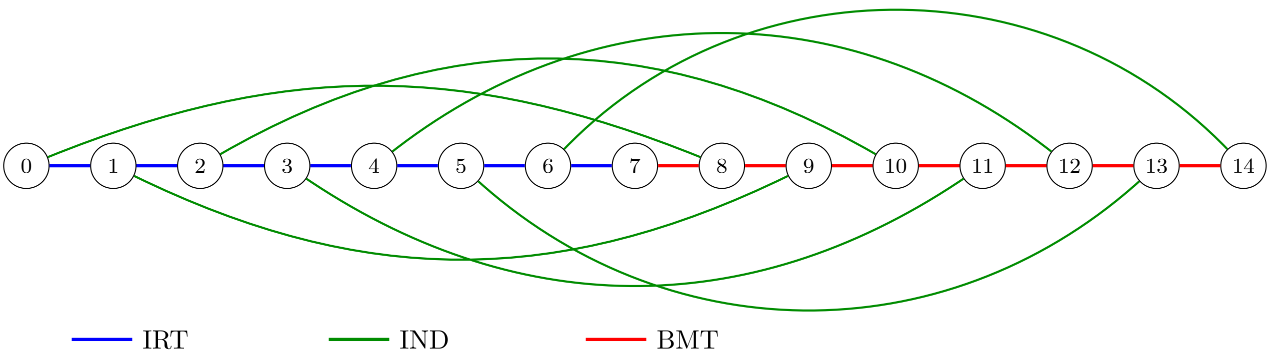
$$7+7+7=21$$

connections.

If the IND shuts down, the IRT and BMT connections form one long path:

$$0-1-2-3-4-5-6-7-8-9-10-11-12-13-14.$$

So the city is connected.



If the IRT shuts down, the BMT connections connect stations 7 through 14, and the IND connections connect stations 0 through 6 into that connected block.

If the BMT shuts down, the IRT connections connect stations 0 through 7, and the IND connections connect stations 8 through 14 into that connected block.

Thus 21 connections are both necessary and sufficient.

$$\boxed{21}$$

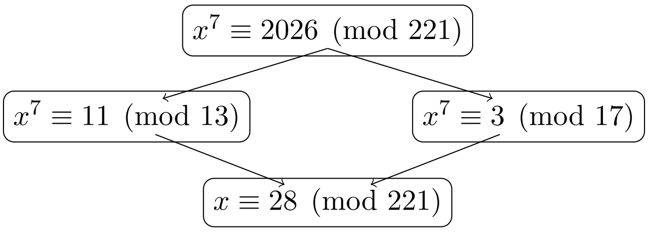
Problem 3. (By Viktor Krapivin, NYCMath) Finance Fred buys his 5th Frolex watch. Tbh, he can't even read the time on it. Its totally messed up with like 221 hour markings around a circle. In an attempt to culture himself Fred decides to learn about alternative systems of time. Inspired by this he riddles you the following problem: Find the smallest positive integer x such that x^7 divided by 221 has the same remainder as 2026 divided by 221. Or put more technically, find the smallest positive integer x such that

$$x^7 \equiv 2026 \mod 221$$

Translation. The modulus factors as

$$221 = 13 \cdot 17.$$

So we solve the congruence modulo 13 and modulo 17, then combine the two answers using the Chinese Remainder Theorem.



Solution. First reduce the right side:

$$2026 = 221 \cdot 9 + 37,$$

so

$$2026 \equiv 37 \pmod{221}.$$

Thus we need

$$x^7 \equiv 37 \pmod{221}.$$

Now factor:

$$221 = 13 \cdot 17.$$

Modulo 13, we have

$$37 \equiv 11 \pmod{13}.$$

So

$$x^7 \equiv 11 \pmod{13}.$$

Among nonzero residues modulo 13, powers repeat with period 12. Since

$$7 \cdot 7 = 49 \equiv 1 \pmod{12},$$

raising both sides to the 7th power undoes the 7th power. Therefore

$$x \equiv 11^7 \pmod{13}.$$

Since $11 \equiv -2 \pmod{13}$,

$$11^7 \equiv (-2)^7 = -128 \equiv 2 \pmod{13}.$$

So

$$x \equiv 2 \pmod{13}.$$

Modulo 17, we have

$$37 \equiv 3 \pmod{17}.$$

So

$$x^7 \equiv 3 \pmod{17}.$$

Again,

$$7 \cdot 7 = 49 \equiv 1 \pmod{16},$$

so

$$x \equiv 3^7 \pmod{17}.$$

Compute:

$$3^2 = 9, \qquad 3^4 = 81 \equiv 13 \pmod{17}.$$

Thus

$$3^7 = 3^4 \cdot 3^2 \cdot 3 \equiv 13 \cdot 9 \cdot 3 = 351 \equiv 11 \pmod{17}.$$

So

$$x \equiv 11 \pmod{17}.$$

Now solve

$$x \equiv 2 \pmod{13},$$
$$x \equiv 11 \pmod{17}.$$

Write

$$x = 2 + 13k.$$

Then

$$2 + 13k \equiv 11 \pmod{17},$$

so

$$13k \equiv 9 \pmod{17}.$$

The inverse of 13 modulo 17 is 4, because

$$13 \cdot 4 = 52 \equiv 1 \pmod{17}.$$

Thus

$$k \equiv 9 \cdot 4 = 36 \equiv 2 \pmod{17}.$$

The smallest positive choice is $k = 2$, giving

$$x = 2 + 13 \cdot 2 = 28.$$

Therefore the answer is

$$\boxed{28}.$$

Problem 4. The price of Bigcoin is a chart that never goes down and takes on integer values each day for 4 days (so you can represent the chart as a 4 digit positive integer). The SEC would like your help to compute all the possible charts that Bigcoin can take on. Said more concretely, how many four-digit positive integers have digits in nondecreasing order from left to right?

Translation. A number like 2337 is allowed because the digits never go down. A number like 2317 is not allowed because $3 > 1$.

$$\boxed{a}\ \boxed{b}\ \boxed{c}\ \boxed{d}$$

$$1 \le a \le b \le c \le d \le 9$$

Solution. Let the digits be

$$a \leq b \leq c \leq d.$$

Because the number is four-digit, the first digit cannot be 0. Since the digits are nondecreasing, once $a \geq 1$, all the later digits are also at least 1.

So we are choosing 4 digits from

$$1, 2, 3, \ldots, 9$$

with repetition allowed. Once the four digits are chosen, their order is forced: they must be written from smallest to largest.

Therefore we can use a stars and bars style argument for this. Imagine we have the digits $(1, 2, 3, 4...9$. Now suppose I put down 4 bars into the list at any point to the left of the 1. Such as $1, |, 2, |, |, 3, 4, 5, 6, 7, 8, 9, |$. This encodes a non decreasing sequence by being read as follows: "the first first digit to the left of a sequence of consecutive bars is the digit in the number at the position of the bar. So in the above configuration the number is 1229. We now wish to count these arrangements. Since the 1 is fixed we don't consider it. So there are 12 symbols in $2, 3, 4, 5, 6, 7, 8, 9, |, |, |$ but the ordering of the numbers is fixed and the bars are indistinguishable so we are really just looking for all the arrangements of $n, n, n, n, n, n, n, n, |, |, |, |$. This is reasoned to be $\frac{12!}{8!4!} = \frac{12*11*10*9}{4*3*2*1} = 11 * 5 * 9 = 495$

A more sophisticated argument is to realize this is the number of multisets of size 4 chosen from 9 digit types and to directly use the binomial expression

$$\binom{9+4-1}{4} = \binom{12}{4} = \frac{12!}{8!4!} = 495.$$

This can be elaborated on by noting that Therefore the answer is

$$\boxed{495}.$$

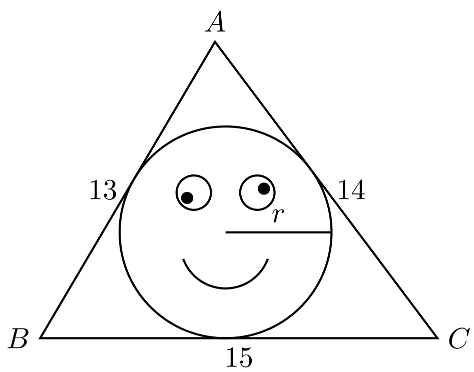
This can be elaborated on by noting that

Problem 5. In the diagram below (not drawn to scale), A circle is tangent to all three sides of the triangle. The side lengths of the triangle are 13, 14, and 15. What is the radius of the circle?

Translation. A circle tangent to all three sides of a triangle is called the incircle. Its radius r satisfies

$$\text{Area} = rs,$$

where s is the semiperimeter.



Solution. The semiperimeter is

$$s = \frac{13+14+15}{2} = \frac{42}{2} = 21.$$

Now use Heron's formula to find the area:

$$K = \sqrt{s(s-13)(s-14)(s-15)}.$$

Substitute $s = 21$:

$$K = \sqrt{21(21-13)(21-14)(21-15)}.$$

So

$$K = \sqrt{21 \cdot 8 \cdot 7 \cdot 6}.$$

Since

$$21 \cdot 8 \cdot 7 \cdot 6 = 7056,$$

we get

$$K = \sqrt{7056} = 84.$$

The inradius formula is

$$K = rs.$$

Thus

$$84 = 21r.$$

Therefore

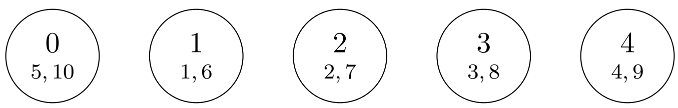
$$r = 4.$$

So the answer is

$$\boxed{4}.$$

Problem 6. Divisibility Dan has been summoned from his secret base under the G train. He riddles you the following: how many subsets of $\{1, 2, 3, \ldots, 10\}$ have sum divisible by 5? (note the empty set is a valid subset and its sum is 0)

Translation. Only remainders modulo 5 matter. The numbers 5 and 10 have remainder 0, so including or excluding them does not change the subset sum modulo 5.



residue classes modulo 5

Solution. The numbers $1, 2, \ldots, 10$ split into residue classes modulo 5:

remainder mod 5	numbers
0	5, 10
1	1, 6
2	2, 7
3	3, 8
4	4, 9

The numbers 5 and 10 do not affect the sum modulo 5. Each may be either included or not included, giving

$$2^2 = 4$$

choices.

Now consider the remaining eight numbers:

$$1, 2, 3, 4, 6, 7, 8, 9.$$

Their residues are

$$1, 1, 2, 2, 3, 3, 4, 4 \pmod 5.$$

We now consider the following procedure. We want to compute "the modulo class of every subset that we have considered so far". We initally start with just empty set and this has sum 0.

So we generate a table:

Keeping track of the subset sums modulo 5, these eight numbers give:

sum mod 5	0	1	2	3	4
number of subsets with only empty set	1	0	0	0	0

Now we want to consider subsets that *could* contain the number 1. We consider all the subsets up to this point and we have the choice to include the number 1 or not. We begin by not including the number 1:

sum mod 5	0	1	2	3	4
number of subsets with only empty set	1	0	0	0	0
number of subsets that could include a 1	1	0	0	0	0

Now we need to consider the sets that COULD include a number 1. If a set has sum that is $0 \mod 5$ and we include a 1 then the new set will be a $1 \mod 5$. So we write:

sum mod 5	0	1	2	3	4
number of subsets with only empty set	1	0	0	0	0
number of subsets that could include a 1	1	$0+1$	0	0	0

And then simplify:

sum mod 5	0	1	2	3	4
number of subsets with only empty set	1	0	0	0	0
number of subsets that could include a 1	1	1	0	0	0

Now lets go after that 6 (which is still 1 mod 5). We begin by consider all the set that could include a 1, and don't have the 6.

sum mod 5	0	1	2	3	4
number of subsets with only empty set	1	0	0	0	0
number of subsets that could include a 1	1	1	0	0	0
number of subsets that could include a 1,6	1	1	0	0	0

Now we add in the number of sets that could include a 1, and DO have a 6.

sum mod 5	0	1	2	3	4
number of subsets with only empty set	1	0	0	0	0
number of subsets that could include a 1	1	1	0	0	0
number of subsets that could include a 1,6	1	$1+1$	$0+1$	0	0

Now we simplify:

sum mod 5	0	1	2	3	4
number of subsets with only empty set	1	0	0	0	0
number of subsets that could include a 1	1	1	0	0	0
number of subsets that could include a 1,6	1	2	1	0	0

Now we pick up our 2. We consider all the number that could include a 1,6 and do NOT have a 2.

sum mod 5	0	1	2	3	4
number of subsets with only empty set	1	0	0	0	0
number of subsets that could include a 1	1	1	0	0	0
number of subsets that could include a 1,6	1	2	1	0	0
number of subsets that could include a 1,6,2	1	2	1	0	0

And now we consider all the subsets that Do have a 2.

sum mod 5	0	1	2	3	4
number of subsets with only empty set	1	0	0	0	0
number of subsets that could include a 1	1	1	0	0	0
number of subsets that could include a 1,6	1	2	1	0	0
number of subsets that could include a 1,6,2	1	2	$1+1$	$0+2$	$0+1$

And we simplify:

sum mod 5	0	1	2	3	4
number of subsets with only empty set	1	0	0	0	0
number of subsets that could include a 1	1	1	0	0	0
number of subsets that could include a 1,6	1	2	1	0	0
number of subsets that could include a 1,6,2	1	2	2	2	1

From here the pattern is established. We consider the 7 (which is also 2 mod 5) (this causes some wrap around! be careful), and continue our way building up this table. The final row after considering everything

sum mod 5	0	1	2	3	4
number of subsets with only empty set	1	0	0	0	0
number of subsets that could include a 1	1	1	0	0	0
number of subsets that could include $1, 6$	1	2	1	0	0
number of subsets that could include $1, 6, 2$	1	2	2	2	1
number of subsets that could include $1, 6, 2, 7$	3	3	3	4	3
number of subsets that could include $1, 6, 2, 7, 3$	6	7	6	7	6
number of subsets that could include $1, 6, 2, 7, 3, 8$	12	14	12	13	13
number of subsets that could include $1, 6, 2, 7, 3, 8, 4$	26	26	25	26	25
number of subsets that could include $1, 6, 2, 7, 3, 8, 4, 9$	52	51	51	51	51

That final 52 in the 0 column counts our subsets. Thus 52 subsets of these eight numbers have sum divisible by 5. Including the independent choices for 5 and 10, the final count is

$$52 \cdot 4 = 208.$$

Therefore the answer is

$$\boxed{208}.$$

In the alternative, we can use a *roots-of-unity filter*. It encodes subset sums as exponents of a polynomial and then isolates exactly the exponents divisible by 5.

First consider the product

$$P(x) = (1 + x^1)(1 + x^2)(1 + x^3) \cdots (1 + x^{10}),$$

with one factor for each number 1 through 10. Expanding it, each factor contributes either its 1 (the number is left out) or its x^k (the number is included). So every term of the expansion corresponds to a subset, and because $x^a \cdot x^b = x^{a+b}$, the exponent of that term equals the sum of the subset. For instance, choosing x^2, x^3, and x^5 gives $x^{2+3+5} = x^{10}$.

Collecting like terms, write

$$P(x) = \sum_s c_s \, x^s,$$

where c_s is the number of subsets whose elements sum to s. The quantity we want is

$$N = \sum_{5|s} c_s \;=\; c_0 + c_5 + c_{10} + \cdots .$$

Setting $x = 1$ collapses every x^s to 1, giving $P(1) = 2 \cdot 2 \cdots 2 = 2^{10} = 1024$, the total over all subsets. This counts everything, so we need a way to keep only the exponents divisible by 5.

A *fifth root of unity* is a number satisfying $w^5 = 1$. Fix a *primitive* one — a root w with $w \neq 1$. Two properties are all that we use:

- **Property 1:** $w^5 = 1$.
- **Property 2:** $w \neq 1$.

The five fifth roots of unity are then $1,\ w,\ w^2,\ w^3,\ w^4$; four of them are complex numbers. By Property 1 the powers of w repeat with period 5 (so $w^5 = 1$, $w^6 = w$, and so on).

The central identity is

$$1 + w + w^2 + w^3 + w^4 = 0.$$

To prove it, expand the product

$$(w-1)(w^4+w^3+w^2+w+1) \;=\; \underbrace{w^5 + w^4 + w^3 + w^2 + w}_{\text{from } w} \;-\; \underbrace{(w^4 + w^3 + w^2 + w + 1)}_{\text{from } -1} \;=\; w^5 - 1.$$

By Property 1 the right-hand side is $1 - 1 = 0$. By Property 2 the factor $w - 1$ is nonzero, so the other factor must vanish: $w^4 + w^3 + w^2 + w + 1 = 0$.

For any integer s,

$$\frac{1}{5}\sum_{k=0}^{4} \left(w^k\right)^s = \frac{1}{5}\left(1^s + w^s + w^{2s} + w^{3s} + w^{4s}\right) = \begin{cases} 1 & \text{if } 5 \mid s, \\ 0 & \text{otherwise.} \end{cases}$$

If $5 \mid s$: each $w^{ks} = (w^5)^{ks/5} = 1$, so the five terms sum to 5 and the average is 1.

If $5 \nmid s$: set $u = w^s$. Then $u^5 = (w^5)^s = 1$, and $u \neq 1$ because w is primitive and $5 \nmid s$. Since $w^{ks} = u^k$, the sum is $1 + u + u^2 + u^3 + u^4 = 0$ by the identity of Step 2, so the average is 0.

Geometric interpretation The five fifth roots of unity are equally spaced points on the unit circle in the complex plane, so their sum (their centroid) is the center of the circle, 0. Raising each to the power s sends them to another equally spaced configuration, still summing to 0 — except when $5 \mid s$, where all five collapse onto the point 1 and sum to 5.

Recall from Step 1 that the answer is the sum of the coefficients of P at the exponents divisible by 5:

$$N = \sum_{5|s} c_s = c_0 + c_5 + c_{10} + \cdots .$$

This sum runs only over the multiples of 5. The first move is to rewrite it as a sum over *all* s, by attaching to each term the filter value from Step 3. Write

$$g(s) = \frac{1}{5}\sum_{k=0}^{4} w^{ks}, \qquad \text{so that } g(s) = 1 \text{ when } 5 \mid s \text{ and } g(s) = 0 \text{ otherwise.}$$

Multiplying c_s by $g(s)$ keeps the term when $5 \mid s$ and replaces it with 0 when $5 \nmid s$. So inserting the factor $g(s)$ into every term lets the index s range over all values while leaving the total unchanged — the only surviving terms are exactly those with $5 \mid s$:

$$N = \sum_{5|s} c_s \;=\; \sum_{s} c_s\, g(s) \;=\; \sum_{s} c_s \left[\frac{1}{5}\sum_{k=0}^{4} w^{ks}\right].$$

The right-hand side is a double sum: an outer sum over s and an inner sum over k. Both are finite (the largest possible subset sum is $1 + 2 + \cdots + 10 = 55$), so their order may be swapped. Doing so, and using $w^{ks} = (w^k)^s$ together with $\sum_s c_s(w^k)^s = P(w^k)$, gives

$$N = \frac{1}{5}\sum_{k=0}^{4}\sum_{s} c_s\left(w^k\right)^s = \frac{1}{5}\sum_{k=0}^{4} P(w^k) = \frac{1}{5}\Big(P(1) + P(w) + P(w^2) + P(w^3) + P(w^4)\Big).$$

It remains to evaluate the five terms. First, $P(1) = 1024$.

For $k = 1, 2, 3, 4$: since $w^5 = 1$, the value w^{ka} depends only on $a \bmod 5$. As a runs over $1, 2, \ldots, 10$, the residues $0, 1, 2, 3, 4$ each occur exactly twice; and because k is coprime to 5, the products ka also cover each residue exactly twice. Hence

$$P(w^k) = \prod_{a=1}^{10}(1 + w^{ka}) = \left(\prod_{r=0}^{4}(1 + w^r)\right)^2.$$

To evaluate the inner product, use the factorization $x^5 - 1 = \prod_{r=0}^{4}(x - w^r)$ and substitute $x = -1$:

$$(-1)^5 - 1 = -2 = \prod_{r=0}^{4}(-1 - w^r) = -\prod_{r=0}^{4}(1 + w^r), \qquad \text{so} \qquad \prod_{r=0}^{4}(1 + w^r) = 2.$$

Therefore $P(w^k) = 2^2 = 4$ for each $k = 1, 2, 3, 4$.

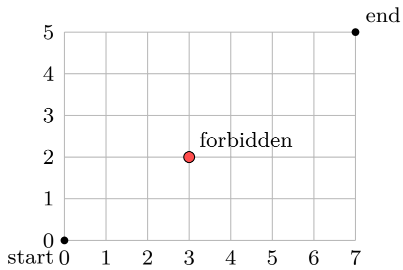
$$N = \frac{1}{5}\big(1024 + 4 + 4 + 4 + 4\big) = \frac{1040}{5} = 208.$$

Therefore the final answer is

$$\boxed{208}.$$

Problem 7. Jay-Z and Euclid were having some difficulties with their time machine and got teleported to Chelsea right during the Knicks playoffs. They wanted to get to Grand Central and there is no Taxi that is willing to drive them so they mus walk. Euclid is claustrophobic and so would prefer to avoid Madison Square Garden. Help them by computing how many lattice paths there from $(0, 0)$ to $(7, 5)$, using only steps right and up, do not pass through $(3, 2)$.

Translation. First count all paths. Then subtract the paths that pass through the forbidden point $(3, 2)$.



Solution. A path from $(0, 0)$ to $(7, 5)$ uses:

$$7$$

right steps and

$$5$$

up steps.

So it has 12 total steps, and we choose which 5 of them are up steps:

$$\binom{12}{5} = 792.$$

Now count paths that pass through $(3, 2)$.

From $(0, 0)$ to $(3, 2)$, a path uses 3 right steps and 2 up steps, so there are

$$\binom{5}{2} = 10$$

ways.

From $(3, 2)$ to $(7, 5)$, a path uses 4 right steps and 3 up steps, so there are

$$\binom{7}{3} = 35$$

ways.

Therefore the number of paths passing through $(3, 2)$ is

$$10 \cdot 35 = 350.$$

So the number of paths that do not pass through $(3, 2)$ is

$$792 - 350 = 442.$$

Thus the answer is

$$\boxed{442}.$$

Problem 8. Optimus Prime visit's Delmonico's to Order some Prime Rib. The staff at Delmonico's tell Optimus that he can have free refills if he can answer this question about Number

Theory. How many positive integers $n \leq 1000$ have exactly 12 positive divisors and are not divisible by 5?

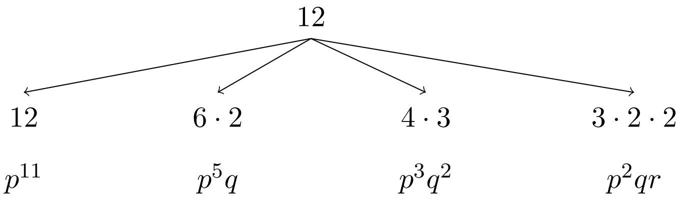
Translation. Use the divisor-counting formula. If

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k},$$

then the number of divisors is

$$(e_1 + 1)(e_2 + 1) \cdots (e_k + 1).$$

We need this product to equal 12.



Solution. The possible exponent patterns are:

$$11, \qquad 5,1, \qquad 3,2, \qquad 2,1,1.$$

These correspond to:

$$p^{11}, \qquad p^5 q, \qquad p^3 q^2, \qquad p^2 q r,$$

where p, q, r are distinct primes.

We also require that n is not divisible by 5, so none of the primes used may be 5.

Case 1: $n = p^{11}$**.**

The smallest possibility is

$$2^{11} = 2048 > 1000.$$

So this case gives 0 numbers.

Case 2: $n = p^5 q$**.**

If $p = 2$, then

$$2^5 q = 32q \leq 1000,$$

so

$$q \leq 31.$$

The possible primes $q \neq 2, 5$ are

$$3, 7, 11, 13, 17, 19, 23, 29, 31.$$

This gives 9 choices.

If $p = 3$, then

$$3^5 q = 243q \leq 1000,$$

so

$$q \leq 4.$$

The only possible prime $q \neq 3, 5$ is

$$q = 2.$$

This gives 1 choice.

If $p \geq 7$, then $p^5 > 1000$. So this case gives

$$9 + 1 = 10.$$

Case 3: $n = p^3q^2$**.**

If $p = 2$, then

$$8q^2 \leq 1000,$$

so

$$q^2 \leq 125.$$

Thus

$$q \leq 11.$$

The possible primes $q \neq 2, 5$ are

$$3, 7, 11.$$

This gives 3 choices.

If $p = 3$, then

$$27q^2 \leq 1000,$$

so

$$q^2 \leq 37.$$

The only possible prime $q \neq 3, 5$ is

$$q = 2.$$

This gives 1 choice.

If $p \geq 7$, then even

$$7^3 \cdot 2^2 = 1372 > 1000.$$

So this case gives

$$3 + 1 = 4.$$

Case 4: $n = p^2qr$**.**

Here p, q, r are distinct primes, none equal to 5. The prime p is the squared prime.

If $p = 2$, then

$$4qr \leq 1000,$$

so

$$qr \leq 250.$$

The valid pairs $q < r$, excluding 2 and 5, give 32 choices.

If $p = 3$, then

$$9qr \leq 1000,$$

so

$$qr \leq 111.$$

The valid pairs give 15 choices.

If $p = 7$, then

$$49qr \leq 1000,$$

so

$$qr \leq 20.$$

Only $(q, r) = (2, 3)$ works. This gives 1 choice.

If $p = 11$, then

$$121qr \leq 1000,$$

so

$$qr \leq 8.$$

Again only $(q, r) = (2, 3)$ works. This gives 1 choice.

If $p \geq 13$, then even

$$13^2 \cdot 2 \cdot 3 = 1014 > 1000.$$

So no more work.

This case gives

$$32 + 15 + 1 + 1 = 49.$$

Adding all cases:

$$0 + 10 + 4 + 49 = 63.$$

Therefore the answer is

$$\boxed{63}.$$

Problem 9. (By Terence Coelho, NYCMath) Squidward and Patrick are on a game show, where there are 10 doors labeled $1, 2, 3, \ldots, 10$. Exactly one of these doors contains \$10,000, and this door is chosen uniformly at random. When the game starts, Squidward and Patrick will each secretly choose a door number. If either of them chose the correct door, the game will end and the player who chose the door with the prize will take home the \$10,000, sharing it between them if they both chose the correct door. Otherwise, they will continue, not learning what door the other just chose.

Squidward knows Patrick extremely well (in particular he's not very bright) and is certain that Patrick will choose doors $1, 2, 3, \ldots$ in that order until the game ends. If he chooses to exploit this and maximize his expected reward from the game, what would his expected reward be?

Translation. Patrick is checking doors in order. Squidward wants to stay one step ahead of him. Instead of starting at door 1, he should start at door 2.

Round	1	2	3	$\cdots$	9
Patrick	1	2	3	$\cdots$	9
Squidward	2	3	4	$\cdots$	10

Squidward wins doors 2 through 10

Solution. Patrick's strategy is fixed:

$$1, 2, 3, \ldots, 10.$$

Squidward wants to choose doors before Patrick gets to them. If he chooses

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 1,$$

then he wins whenever the prize is behind one of the doors

$$2, 3, 4, 5, 6, 7, 8, 9, 10.$$

That is 9 doors out of 10.

If the prize is behind door 1, Patrick wins immediately on the first round, and Squidward gets nothing.

So Squidward gets \$10,000 with probability

$$\frac{9}{10}.$$

His expected reward is therefore

$$\frac{9}{10} \cdot 10000 = 9000.$$

Thus the answer is

$$\boxed{9000}.$$

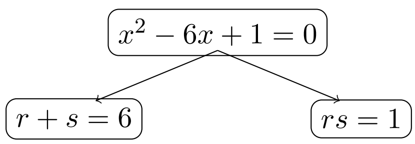
Problem 10. Isaac Newton himself (this looks like a hint if you don't need a hint) has a hard math problem. Let r and s be the two roots of

$$x^2 - 6x + 1 = 0.$$

Find

$$r^4 + s^4.$$

Translation. We do not need to solve for r and s. Vieta's formulas give $r+s$ and rs, and those are enough.



Solution. By Vieta's formulas,

$$r + s = 6$$

and

$$rs = 1.$$

First compute

$$r^2 + s^2.$$

We have

$$r^2 + s^2 = (r+s)^2 - 2rs.$$

So

$$r^2 + s^2 = 6^2 - 2(1) = 36 - 2 = 34.$$

Now compute

$$r^4 + s^4.$$

Using the same trick,

$$r^4 + s^4 = (r^2 + s^2)^2 - 2r^2s^2.$$

Since

$$r^2s^2 = (rs)^2 = 1,$$

we get

$$r^4 + s^4 = 34^2 - 2 = 1156 - 2 = 1154.$$

Thus the answer is

$$\boxed{1154}.$$

Problem 11. One day you suddenly wake up in the matrix... except its unfortunately the Matrix Classic which features 0.1p low res graphics. You look around you and the world is a grid of glowing orbs upon a dark empty void of a background. At every integer coordinate is an orb. You're trying to figure out where you are "supposed" to be and so you start by measuring your room's volume. To do this you will count the orbs that you see which happens to be equivalent to the following problem:

How many ordered triples (a, b, c) of integers satisfy

$$a + b + c = 0$$

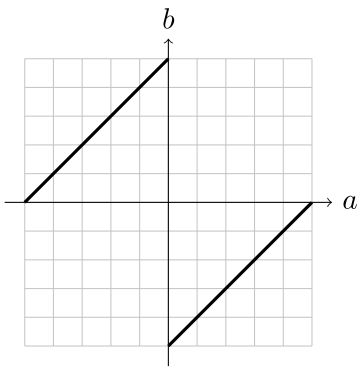
and

$$-5 \le a, b, c \le 5?$$

Translation. Once a and b are chosen, c is forced:

$$c = -a - b.$$

So we count pairs (a, b) for which c also lies between -5 and 5.



valid band: $-5 \le a + b \le 5$

Solution. Once a and b are chosen, c is forced:

$$c = -a - b.$$

So we need

$$-5 \le -a - b \le 5.$$

Equivalently,

$$-5 \le a + b \le 5.$$

There are 11 possible values for a and 11 possible values for b, so there are

$$11 \cdot 11 = 121$$

total ordered pairs (a, b).

Now subtract the bad pairs.

The pairs with

$$a + b = 6, 7, 8, 9, 10$$

come in counts

$$5, 4, 3, 2, 1.$$

So there are

$$5 + 4 + 3 + 2 + 1 = 15$$

pairs with $a + b > 5$.

By symmetry, there are also 15 pairs with $a + b < -5$.

Thus the valid number is

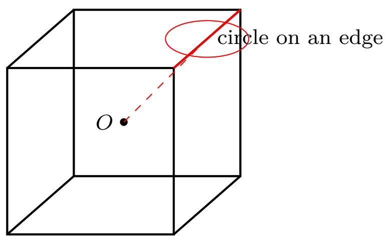
$$121 - 15 - 15 = 91.$$

Therefore the answer is

$$\boxed{91}.$$

Problem 12. Aliens descend upon NYC and decide to park their ship directly above the Astor Place Cube in the East village. A giant laser shoots down and suddenly the cube starts to glow! For each edge of the cube the laser creates a glowing circle with the edge as the diameter. The circle is also perpendicular to the line from the center of the cube to the midpoint of the edge. In this way the aliens constructed 12 circles. At how many points do at least two circles intersect?

Translation. We interpret this as follows: for each edge, the circle has that edge as a diameter and lies in the plane perpendicular to the line from the cube's center to the edge midpoint.



Solution. Place the cube in coordinates with vertices $(\pm 1, \pm 1, \pm 1)$ and center $(0, 0, 0)$.

Consider one edge, for example the edge $x = 1$, $y = 1$, $-1 \leq z \leq 1$. Its midpoint is $(1, 1, 0)$, so the line from the center of the cube to this midpoint points in direction $(1, 1, 0)$. Therefore the circle corresponding to this edge lies in the plane perpendicular to $(1, 1, 0)$ and passing through $(1, 1, 0)$. That plane is

$$(1, 1, 0) \cdot (x - 1, y - 1, z) = 0,$$

which simplifies to $(x - 1) + (y - 1) = 0$, or $x + y = 2$.

The radius of the circle is 1, because the edge length of the cube is 2. Thus this circle has center $(1, 1, 0)$ and radius 1, so its equation is $(x - 1)^2 + (y - 1)^2 + z^2 = 1$. Together, the first circle is described by

$$\boxed{x + y = 2} \quad \text{and} \quad \boxed{(x - 1)^2 + (y - 1)^2 + z^2 = 1}.$$

Now consider an adjacent edge, for example $x = 1$, $z = 1$, $-1 \le y \le 1$. Its midpoint is $(1, 0, 1)$, so the line from the origin to this midpoint points in direction $(1, 0, 1)$. The corresponding circle lies in the plane perpendicular to $(1, 0, 1)$ through $(1, 0, 1)$, namely

$$(1, 0, 1) \cdot (x - 1, y, z - 1) = 0,$$

which simplifies to $(x - 1) + (z - 1) = 0$, so $x + z = 2$. This circle also has radius 1, so its equation is $(x - 1)^2 + y^2 + (z - 1)^2 = 1$. Together, the second circle is described by

$$\boxed{x + z = 2} \qquad \text{and} \qquad \boxed{(x - 1)^2 + y^2 + (z - 1)^2 = 1}.$$

Now solve the equations for the intersection of these two circles. From the two plane equations $x + y = 2$ and $x + z = 2$, we get $y = 2 - x$ and $z = 2 - x$.

Substitute these into the first circle equation $(x - 1)^2 + (y - 1)^2 + z^2 = 1$. Since $y = 2 - x$ and $z = 2 - x$,

$$(x - 1)^2 + \big((2 - x) - 1\big)^2 + (2 - x)^2 = 1,$$

that is, $(x-1)^2+(1-x)^2+(2-x)^2 = 1$. Since $(1-x)^2 = (x-1)^2$, this becomes $2(x-1)^2+(2-x)^2 = 1$. Expanding,

$$\begin{aligned} 2(x^2 - 2x + 1) + (x^2 - 4x + 4) &= 1, \\ 2x^2 - 4x + 2 + x^2 - 4x + 4 &= 1, \\ 3x^2 - 8x + 6 &= 1, \\ 3x^2 - 8x + 5 &= 0, \\ (3x - 5)(x - 1) &= 0. \end{aligned}$$

So $x = 1$ or $x = \frac{5}{3}$.

If $x = 1$, then $y = 2 - 1 = 1$ and $z = 2 - 1 = 1$, so one intersection point is $(1, 1, 1)$. This is a corner of the cube.

If $x = \frac{5}{3}$, then $y = 2 - \frac{5}{3} = \frac{1}{3}$ and $z = 2 - \frac{5}{3} = \frac{1}{3}$, so the other intersection point is $\left(\frac{5}{3}, \frac{1}{3}, \frac{1}{3}\right)$. This point is outside the cube.

Thus, for this particular pair of adjacent edges, the two circle intersections are $(1, 1, 1)$ and $\left(\frac{5}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

The first point is the cube corner shared by the two edges and the second is outside the cube, on the side of the face containing the two edges.

What about non-adjacent edges? If the midpoints are distance greater than 2 apart, the circles cannot intersect as they have radius 1. If the midpoints are distance exactly two, they could only intersect if they lie in the same plane. This eliminates all pairs of non-adjacent edges.

Consider now two pairs of adjacent edges. Can they have the same point of intersection outside the cube? In order for this to be the case, they would need to all lie on the same face of the cube, as the outside point is on the side of the face that the adjacent edges lie on. But two distinct pairs of adjacent edges on the same face cannot all be pairwise adjacent so they cannot all have a simultaneous point of intersection by the preceding paragraph. So this point of intersection outside the cube is unique to a given pair of adjacent edges.

Now count the intersections. At every corner of the cube, exactly 3 edges meet. From those 3 edges, the number of pairs of edges is $\binom{3}{2} = 3$. Each pair of adjacent edges gives one cube-corner intersection and one extra outside intersection.

There are 8 corners of the cube. Each corner itself is counted as one intersection point, so the cube corners contribute 8 intersection points. At each corner there are 3 pairs of adjacent edges, and each such pair gives one extra outside point, so the number of extra outside points is $8 \cdot 3 = 24$.

So the total number of intersection points is $8 + 24 = 32$. Equivalently, each corner contributes $1 + 3 = 4$ intersection points: the corner itself, plus 3 extra outside points coming from the 3 pairs of edges meeting there. Since there are 8 corners, the total is $8 \cdot 4 = 32$.

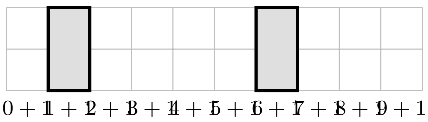
Therefore the answer is

$$\boxed{32}.$$

Problem 13. Sally and the crew were hanging out at Domino park playing Dominoes. A giant soccer ball suddenly hit their game and scattered all the dominoes! The crew was trying to remember their exact configuration but have gotten stuck. The game was played on a 2×10 rectangle with normal 2 units by 1 unit dominoes. Sally did remember that the rectangle was perfectly tiled/covered by non overlapping dominoes and there were exactly 2 vertical dominoes. Can you help her count how many possible configurations could have been in place?

Translation. A vertical domino covers both squares in one column. Once we choose the two vertical domino columns, all remaining blocks must be tiled horizontally. A horizontal-only $2 \times k$ strip works exactly when k is even.

example: vertical dominoes in columns 2 and 7



Solution. Suppose the two vertical dominoes are placed in columns

$$i < j.$$

Then the remaining horizontal-only blocks have lengths

$$i - 1, \qquad j - i - 1, \qquad 10 - j.$$

Each of these must be even.

The condition

$$i - 1 \text{ is even}$$

means i is odd.

The condition

$$10 - j \text{ is even}$$

means j is even.

If i is odd and j is even, then $j - i$ is odd, so

$$j - i - 1$$

is even automatically.

Thus we just need to choose an odd column i and a later even column j.

The odd columns are

$$1, 3, 5, 7, 9.$$

The even columns are

$$2, 4, 6, 8, 10.$$

Count:

i	possible j
1	$2, 4, 6, 8, 10$
3	$4, 6, 8, 10$
5	$6, 8, 10$
7	$8, 10$
9	10

So the total is

$$5 + 4 + 3 + 2 + 1 = 15.$$

Therefore the answer is

$$\boxed{15}.$$

Problem 14. Frodo Baggins is in the town of Rivendell and feeling a little lazy. Instead of walking he has decided to book a flight on Spirit Airlines to the Shire and is trying to pack as much treasure as he can. His approved suitcase can store a total of 1 cubic meter. He has received from his Fellowship infinitely many objects, in fact for every natural number n he has received exactly 3 objects of size $1/n$ cubic meters. He can't pack it all and he wants to waste no space so he needs your help to know: how many distinct ordered triples (a, b, c) of positive integers exactly satisfy

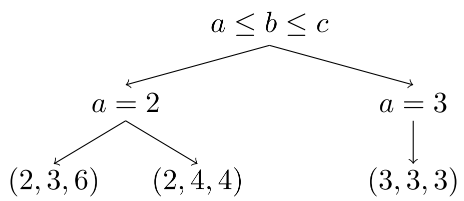
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1?$$

(Fun fact, Frodo's treasure also reveals the volume of Rivendell's Treasure Vault is infinite but that is not a mystery we will concern ourselves on today's math contest).

Translation. This is an Egyptian fraction equation. First sort the triple so that

$$a \le b \le c.$$

Then find the few sorted possibilities. Finally count all orderings.



Solution. First assume

$$a \le b \le c.$$

Then

$$\frac{1}{a} \ge \frac{1}{b} \ge \frac{1}{c}.$$

Since

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1,$$

we must have

$$\frac{3}{a} \ge 1.$$

So

$$a \le 3.$$

Also $a \neq 1$, since $\frac{1}{1} = 1$ would already use the whole sum and the other positive fractions would make the sum too large.

Therefore

$$a = 2$$

or

$$a = 3.$$

Case 1: $a = 2$.

Then

$$\frac{1}{2} + \frac{1}{b} + \frac{1}{c} = 1,$$

so

$$\frac{1}{b} + \frac{1}{c} = \frac{1}{2}.$$

Multiplying by $2bc$, we get

$$2c + 2b = bc.$$

Rearrange:

$$bc - 2b - 2c = 0.$$

Add 4 to both sides:

$$bc - 2b - 2c + 4 = 4.$$

Factor:

$$(b - 2)(c - 2) = 4.$$

Since $b \le c$, the factor pairs are

$$1 \cdot 4, \qquad 2 \cdot 2.$$

Thus

$$(b, c) = (3, 6)$$

or

$$(b, c) = (4, 4).$$

So we get

$$(2, 3, 6), \qquad (2, 4, 4).$$

Case 2: $a = 3$.

Then

$$\frac{1}{3} + \frac{1}{b} + \frac{1}{c} = 1,$$

so

$$\frac{1}{b} + \frac{1}{c} = \frac{2}{3}.$$

Since $b \ge 3$ and $c \ge b$,

$$\frac{1}{b} + \frac{1}{c} \le \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Equality happens only when

$$b = c = 3.$$

So the only solution in this case is

$$(3, 3, 3).$$

The sorted solutions are

$$(2,3,6), \qquad (2,4,4), \qquad (3,3,3).$$

Now count ordered triples.

The triple $(2,3,6)$ has all entries distinct, so it has

$$3! = 6$$

orderings.

The triple $(2,4,4)$ has two equal entries, so it has

$$3$$

orderings.

The triple $(3,3,3)$ has only

$$1$$

ordering.

Therefore the total number of ordered triples is

$$6+3+1=10.$$

So the answer is

$$\boxed{10}.$$

Problem 15. (Proposed by Patrick Chen (Google), originally proposed by Elliot Line in Actually Good Math Problems on Facebook) Bobby and the friends decide to order a pepperoni pizza pie from Empire Slice and Co. When their order is ready the Chef runs in yelling "We have a problem!!!"

The Chef explains: "I thought I was baking a regular pizza but when I pulled it out the pepperoni had formed a face and the pizza came to life! It said 'Go Spurs!' and I was terrified so I accidentally chopped it and now the slice is all wrong. I'll tell you what. If you can guess the product of the areas to the left and right of the slice I made I'll give you the pizza for free. The pizza is a perfect regular nonagon with area 9, the slice in the middle has area 2 and contains exactly one entire side of the nonagon. Let x be the area of the pizza to the left of the slice, and let y be the area of the pizza to the right of the slice, so that $x+y+2=9$. Compute the product xy."

Bobby and the group stare down at the pizza and now look to you for some help. Please tell them the right answer.

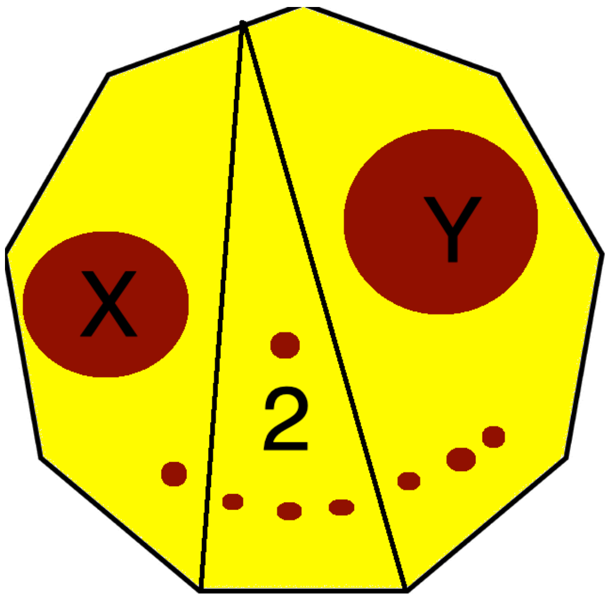


Figure 1: The magical pizza of total area 9

Translation. The whole nonagon has area 9. The middle slice has area 2. Therefore the two remaining pieces have total area

$$x + y = 7.$$

Solution. The total area of the regular nonagon is

$$9.$$

The middle slice has area

$$2.$$

Before we proceed its helpful to consider the following diagram. Lets just consider two slices of that nonagon meeting at the center and bounded by parallel lines. (The lines drawing down are parallel because the nonagon is symmetric about the side labeled L.

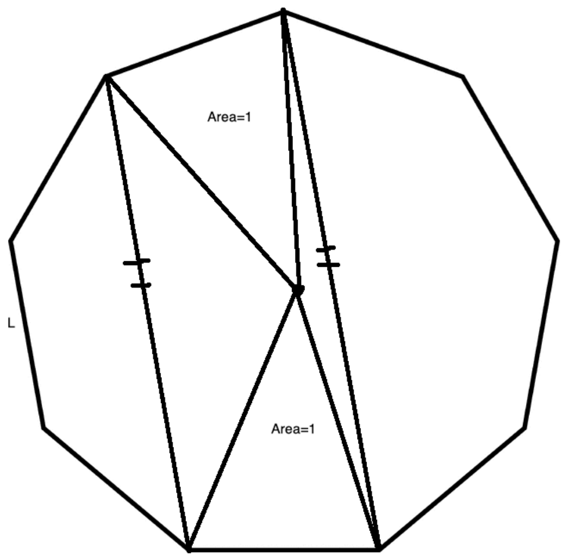


Figure 2: Two of the 9 equal pizza slices that divide the nonagon

The area of the two triangles is 1 each. And combined they form an area of 2. To the left of the triangles the entire area is 3 and to the right the entire area is 4. (This can be made clear by drawing a line segment from every corner of nonagon to the center and seeing that it forms 9 triangles each of area 1. We have also labelled the triangles $T1$ and $T2$. These characters will be important in our story. Now suppose we add an additional dotted line parallel to the already two identified parallel lines.

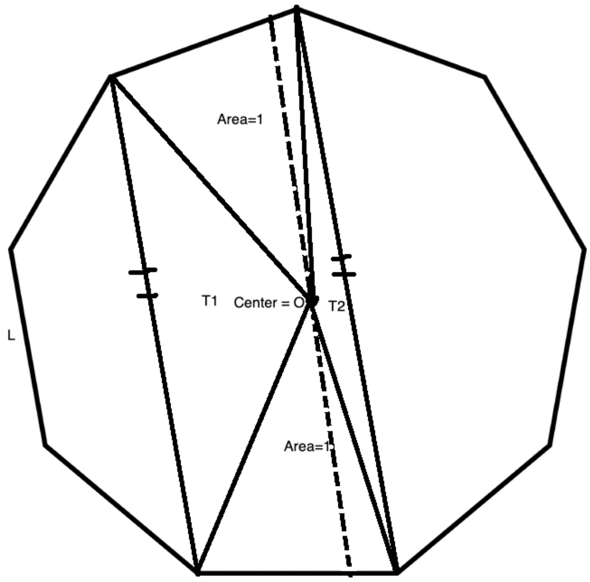


Figure 3: Dotted line is introduced

With some imagination you can imagine sliding the center O along the dotted line, and the upper triangle of area 1 gets smaller while the lower triangle of area 2 gets larger. What's very counterintuitive is the areas of T1 and T2 do NOT change no matter how you go about sliding along that dotted line. This is because the base of both T1 and T2 is fixed as B_1, B_2, and this dotted line (being parallel to the other two parallel lines) induces the exact same height for $T1$ and $T2$ as before we did any sliding. As a result regardless of how we slide along the dotted line the area of $T1 = \frac{1}{2}B_1H_1$ and similarly the area of $T2 = \frac{1}{2}B_2H_2$. See the diagram below:

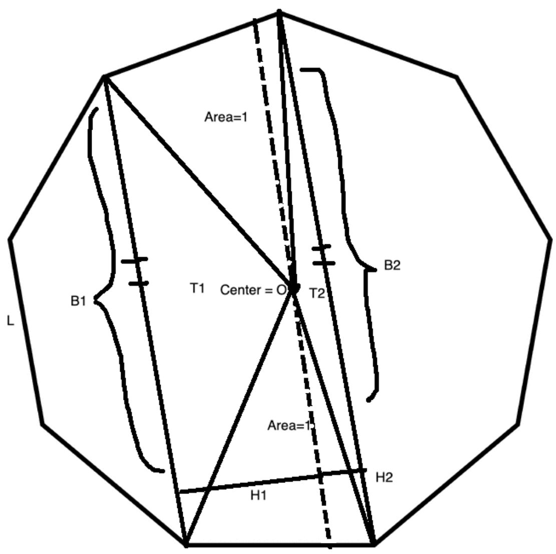


Figure 4: Bases and Heights marked

So if regardless how we slide along the dotted line the $T1$ and $T2$ area doesn't change and then is given that the combined area of the two triangles is always going to be 2. But here's the kicker. We can slide on the dotted line ALL the way up and into the border of nonagon and then we end up with the initial configuration we sought:

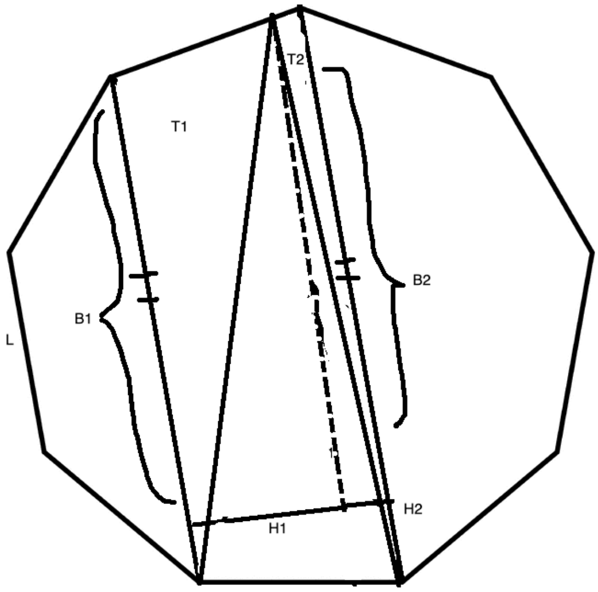


Figure 5: We see that the original configuration is just a special case of sliding on the dotted line all the way

Therefore the area to the left of our triangular slice is always constantly 3 and similar the area to the right is always constantly 4. We find:

$$xy = 3 \cdot 4 = 12.$$

So the answer is

$$\boxed{12}.$$